LAMINAR HEAT TRANSFER IN A TWO-DIMENSIONAL CHANNEL WITH A NONUNIFORM TEMPERATURE FIELD AT THE ENTRANCE

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Heat transfer is investigated, on the basis of the linearized energy equation, in the initial section of a two-dimensional channel with laminar flow of an incompressible fluid and a nonuniform temperature field at the entrance. Results of calculations of the temperature field and Nusselt number Nu are given for various cross sections along the channel.

The design and calculation of heat exchangers frequently involves heat transfer processes in short channels in which hydrodynamically and thermally stabilized flow is not achieved and the working medium reaches the channel nonuniformly heated over the cross section. The heat transfer coefficient and channel wall temperature then depend on the temperature field at the entrance. Let us examine the heat transfer conditions in a two-dimensional channel for a steady flow of an incompressible viscous fluid for which the physical properties c_p and λ are independent of temperature. For large Re numbers, i.e., within the limits of boundary-layer theory, the heat transfer in a two-dimensional channel is described by the energy equation [1]:

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \lambda \frac{\partial^2 T}{\partial y^2}. \tag{1}$$

where the term taking into account energy dissipation in friction has been omitted.

In [2] the flow of a fluid in the entrance region of a tube was studied using the approximate equation of motion $V_0 \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$, in which the term $v\, \partial u/\partial y$ was neglected in view of the smallness of the transverse velocity component, while in the coefficient attached to the derivative $\partial u/\partial x$ the approximation $u = V_0$ was introduced, where V_0 is the mean velocity over the section of the fluid in the tube. The velocity distribution and pressure in the tube obtained by solving this approximate equation agreed well enough with experimental data, and at large distances from the entrance tended to the Poiseuille distribution. A certain difference from the experimental data, observed in [2], is connected with the author's assumption that the term $v\, \partial u/\partial y$ is small compared with $u\, \partial u/\partial x$. At the same time, as follows from the Blasius solution, the ratio of the convection terms in the equation of motion, $(v\, \partial u/\partial y)/(u\, \partial u/\partial x)$, has a nearly constant value of 0.5 in the main part of the boundary layer and only for u/V_0 from 0.85 to 0.99, i.e., near the outer edge of the boundary layer, does it change (from 0.45 to 0.33).

Since the interaction between the boundary layers formed near the upper and lower walls is small in the entrance region of a two-dimensional channel, in solving the energy equation with Pr ~ 1, we are quite justified in considering the ratio of the convective terms to be constant, and in introducing a proportionality constant ε independent of the coordinates.

Then the energy equation may be written in the form

$$\varepsilon V_0 \frac{\partial T}{\partial x} = a \frac{\partial^2 T}{\partial y^2}, \tag{2}$$

where the correction ε takes into account the omitted term $v\, \partial T/\partial y$ and the replacement of the longitudinal velocity component u by its mean value over the channel section V_0, equal to the fluid velocity in the entrance region. It is shown in [3] that $\varepsilon = 0.346\, \mathrm{Pr}^{-1/3}$.

Going over in (2) to the dimensionless variables $\xi = x/h$, $\eta = y/h$, $\mathrm{Pe} = V_0 h/a$ and $\theta = (T_1 - T)/(T_1 - T_0)$, gives

$$\varepsilon \mathrm{Pe} \frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \eta^2}. \tag{3}$$

Suppose that a fluid flow nonuniformly heated over its height enters a two-dimensional channel of height 2h, the fluid temperature being given by the formula

$$T_{bx} = T_0 + b\eta + c\eta^2. \tag{4}$$

Introducing the dimensionless parameters $\beta = b/(T_1 - T_0)$, $\gamma = c/(T_1 - T_0)$, we have

$$\theta_{bx} = 1 - \beta\eta - \gamma\eta^2. \qquad (4')$$

Note that a negative value of the true temperature T enters into the dimensionless temperature θ, so that minimum θ corresponds to maximum T, and vice versa.

Let us make the channel wall temperature a linear function of the longitudinal coordinate $T = T_1 + \Delta T_1 \xi$ for $\eta = \pm 1$. Going over to the dimensionless temperature, we have

$$\eta = \pm 1,\ \theta = -\vartheta\xi, \qquad (5)$$

where $\vartheta = \Delta T_1/(T_1 - T_0)$.

Thus, to find the temperature distribution over the section of a two-dimensional channel with a given wall temperature, Eq. (3) must be solved with initial condition (4) and boundary condition (5).

In essence, Eq. (3) is an equation of unsteady heat conduction, methods of solving which have been developed in detail in [4]. Applying a Laplace transform to (3) and using initial condition (4), we obtain

$$d^2\theta^*/d\eta^2 - s\varepsilon \mathrm{Pe}\,\theta^* + \varepsilon \mathrm{Pe}\,(1 - \beta\eta - \gamma\eta^2) = 0. \qquad (6)$$

Solving this equation and using boundary condition (5), we find θ^*. Expanding the expression for θ^* in a series of exponential functions and then converting from the transform to the original, we obtain the final solution for the temperature distribution inside the channel in terms of probability functions erfc x and integrals of the probability functions i^2 erfc x [4]:

$$\begin{aligned}
\theta = {} & 4\xi\left(\frac{2\gamma}{\varepsilon \mathrm{Pe}} - \nu\right)\sum_{n=1}^{\infty}(-1)^{n+1}\left[\mathrm{i^2erfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \right. \\
& \left. + \mathrm{i^2erfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] + (\gamma - 1)\sum_{n=1}^{\infty}(-1)^{n+1}\times \\
& \times\left[\mathrm{erfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \mathrm{erfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] + \\
& + 1 - \beta\eta - \gamma\eta^2 + \beta\sum_{n=1}^{\infty}\left[\mathrm{erfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) - \right. \\
& \left. - \mathrm{erfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] - \frac{2\gamma}{\varepsilon \mathrm{Pe}}\,\xi.
\end{aligned} \qquad (7)$$

To find the Nu number based on the local difference between the wall temperature and the bulk temperature of the fluid, the heat transfer coefficient must be determined:

$$\alpha = -\lambda\left(\frac{\partial T}{\partial y}\right)_{\mathrm{w}} \Big/ (T_{\mathrm{m}} - T_{\mathrm{w}}). \qquad (8)$$

From this we obtain

$$\mathrm{Nu} = \frac{\alpha h}{\lambda} = -\left(\frac{\partial\theta}{\partial\eta}\right)_{\mathrm{w}} \Big/ (\theta_{\mathrm{m}} - \theta_{\mathrm{w}}). \qquad (9)$$

It may be seen from (9) that to determine the local Nu number the mean gas temperature and the temperature gradient at the wall must be found. We find the latter by differentiating (7) with respect to η. To determine the integral mean fluid temperature, we must integrate the expression for the transform θ^* with respect to η in the interval $\eta = -1$ to $\eta = 1$, and then convert from transform to original. We then obtain

$$\theta_{\mathrm{m}} = 1 - \frac{\gamma}{3} - \frac{2\gamma}{\varepsilon \mathrm{Pe}}\xi + \frac{2(\gamma - 1)\sqrt{\xi}}{\sqrt{\pi\varepsilon \mathrm{Pe}}} - \frac{4(\gamma - 1)\sqrt{\xi}}{\sqrt{\varepsilon \mathrm{Pe}}} \times$$

$$\times \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{ierfc}\left(n \sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \frac{8\gamma\xi^{3/2}}{3\sqrt{\pi}(\varepsilon \mathrm{Pe})^{3/2}} - 16(2\gamma - \vartheta\varepsilon \mathrm{Pe}) \times$$
$$\times \left(\frac{\xi}{\varepsilon \mathrm{Pe}}\right)^{3/2} \sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{i}^3\mathrm{erfc}\left(n \sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right). \tag{10}$$

Using the expressions for the temperature gradient and the mean fluid temperature, we obtain

$$\mathrm{Nu} = A/B, \tag{11}$$

$$-A = 2\xi\left(\frac{2\gamma}{\varepsilon \mathrm{Pe}} - \vartheta\right)\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}} \sum_{n=1}^{\infty} (-1)^{n+1} \left\{\operatorname{ierfc}\left[(n-1)\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right] - \right.$$
$$\left. - \operatorname{ierfc}\left(n\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right\} + (\gamma - 1)\sqrt{\frac{\varepsilon \mathrm{Pe}}{\pi\xi}} \sum_{n=1}^{\infty} (-1)^{n+1} \times$$
$$\times \left\{ \exp\left[-(n-1)^2 \frac{\varepsilon \mathrm{Pe}}{\xi}\right] - \right.$$
$$\left. - \exp\left(-n^2 \frac{\varepsilon \mathrm{Pe}}{\xi}\right)\right\} + \beta\sqrt{\frac{\varepsilon \mathrm{Pe}}{\pi\xi}} \times \tag{11'}$$
$$\times \sum_{n=1}^{\infty} \left\{\exp\left[-(n-1)^2 \frac{\varepsilon \mathrm{Pe}}{\xi}\right] + \right.$$
$$\left. + \exp\left(-n^2 \frac{\varepsilon \mathrm{Pe}}{\xi}\right)\right\},$$

$$B = \theta_{\mathrm{av}} + \vartheta\xi. \tag{11''}$$

Let us apply the relation obtained to heat transfer in short channels in which the flow in the stabilizing section is laminar even for $\mathrm{Re} > 3 \cdot 10^3$. At a certain distance from the entrance, not exceeding $\xi \le 100$, for $\mathrm{Re} \ge 3 \cdot 10^3$ the expression for Nu is much simplified. For Nu at the upper wall ($\eta = 1$) we have

$$\mathrm{Nu}|_{\eta=1} = \frac{\beta + 2\gamma + \sqrt{\dfrac{\varepsilon \mathrm{Pe}}{\pi\xi}}\left(1 - \gamma - \beta + 2\vartheta\xi - \dfrac{4\gamma\xi}{\varepsilon \mathrm{Pe}}\right)}{1 - \dfrac{\gamma}{3} + \vartheta\xi - \dfrac{2\gamma\xi}{\varepsilon \mathrm{Pe}} + 2\sqrt{\dfrac{\xi}{\pi\varepsilon \mathrm{Pe}}}\left[\dfrac{2}{3}\xi\left(\dfrac{2\gamma}{\varepsilon \mathrm{Pe}} - \vartheta\right) + \gamma - 1\right]}. \tag{12}$$

Figure 1 gives the Nu distribution along the upper channel wall ($\eta = 1$) for various values of γ and β. The smaller γ, the lower the fluid temperature near the wall and the more the fluid is heated in the core of the flow. For negative γ the fluid temperature near the wall is less than in the core, while for positive γ the contrary holds true. Negative values of β correspond to the case when the coldest fluid layers are at the upper channel wall, and the most highly heated layers at the lower wall.

It may be seen by examining Fig. 1 that an increase in γ leads to a decrease in Nu at the upper wall ($\eta = 1$) and an increase at the lower wall ($\eta = -1$). There are three cases in which heat transfer conditions at the upper wall will vary, depending on the value of the expression $(1 - \gamma - \beta)$. For $(1 - \gamma - \beta) > 0$ the gas temperature at the wall in the entrance section will be lower than the wall temperature, and heat transfer will take place from the wall to the gas, and the larger the value of $(1 - \gamma - \beta)$ the greater the Nu number at the upper wall (curves 1-7). When $(1 - \gamma - \beta) =$ $= 0$, the gas temperature at the wall in the entrance section is equal to the wall temperature, and the Nu number near the entrance is constant over the channel length, its value depending on the temperature gradient at the wall in the entrance section (curve 8). If $(1-\gamma-\beta) < 0$, the gas temperature at the wall in the entrance section is higher than the wall temperature and heat transfer from the gas to the wall takes place (curves 9, 10).

If heat flux through the walls of a two-dimensional channel is given, the boundary conditions may be written as follows:

$$y = \pm h, \ \frac{\partial T}{\partial y} = \pm \frac{q_0}{\lambda} \pm \frac{q_1 x}{\lambda h}. \tag{13}$$

Let the temperature at the channel entrance be that given by (4). Introducing the dimensionless temperature $\theta_q = (T - T_0)/T_0$, we see that, to determine the temperature distribution in the channel with the heat flux given, it is necessary to solve (3) with the following boundary and initial conditions:

$$\xi = 0, \ \theta_q = \beta_q \eta + \gamma_q \eta^2,$$
$$\eta = \pm 1, \ \partial \theta_q / \partial \eta = \pm K_0 \pm K_1 \xi, \tag{14}$$

where $\beta_q = b/T_0$, $\gamma_q = c/T_0$, $K_0 = q_0 h/\lambda T_0$, $K_1 = q_1 h/\lambda T_0$.

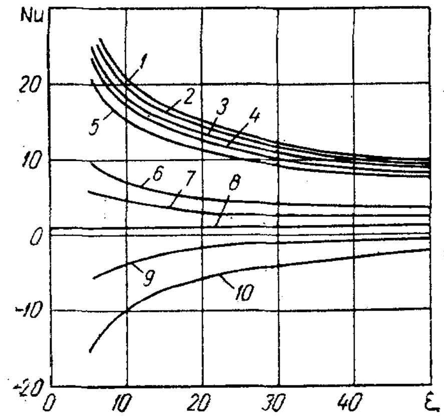

Fig. 1. Variation of local Nu number along a two-dimensional channel with a nonuniform temperature field at the entrance and given wall temperature, $\vartheta = 0$, Pr = 0.7, Re = 10^4. $\beta = -1$: $1-\gamma = (-1)$, 7 – (–0.5), 8 – 0, 9 – 0.5, 10 – 1.

Here the solution of (3) is

$$\theta_q = 2(K_0 - 2\gamma_q)\sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}} \sum_{n=1}^{\infty} \left[\mathrm{ierfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \right.$$
$$\left. + \mathrm{ierfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] + 8K_1 \xi \sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}} \times$$
$$\times \sum_{n=1}^{\infty} \left[\mathrm{i^3erfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) + \mathrm{i^3erfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] - \tag{15}$$
$$-2\beta_q \sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}} \sum_{n=1}^{\infty} (-1)^{n+1} \left[\mathrm{ierfc}\left(\frac{2n-1-\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right) - \right.$$
$$\left. - \mathrm{ierfc}\left(\frac{2n-1+\eta}{2}\sqrt{\frac{\varepsilon \mathrm{Pe}}{\xi}}\right)\right] + \beta_q \eta + \gamma_q \eta^2 + \frac{2\gamma_q}{\varepsilon \mathrm{Pe}} \xi.$$

The Nu number is found in the same way as for the case of a given wall temperature. Finally, for $\xi \leq 100$, Re $\geq 3 \cdot 10^3$ we obtain

$$\mathrm{Nu}|_{\eta=1} = A/B, \tag{16}$$

$$A = K_0 + K_1 \xi, \tag{17}$$

$$B = 2(K_0 - 2\gamma_q)\sqrt{\frac{\xi}{\pi \varepsilon \mathrm{Pe}}} + 0{,}75 \frac{K_1 \xi^{3/2}}{\sqrt{\varepsilon \mathrm{Pe}}} - 2\beta_q \sqrt{\frac{\xi}{\varepsilon \mathrm{Pe}}} -$$
$$- \frac{1}{\varepsilon \mathrm{Pe}}(K_0 \xi + K_1 \xi^2 - 2\gamma_q \xi) + \beta_q + \frac{2}{3}\gamma_q. \tag{18}$$

The variation of Nu with channel length is shown in Fig. 2. The singularity in Nu distribution over the channel length for the case $\gamma_q < 0$ is connected with the fact the mean mass temperature of the fluid near the entrance is lower than the wall temperature, but at a certain distance from the entrance the wall temperature becomes equal to the mean mass temperature, i. e., the denominator in the expression for the heat transfer coefficient $\alpha = \lambda(\partial T/\partial y)_w/(T_m - T_w)$ becomes equal to zero. As the distance from the entrance increases, $(T_m - T_w)$, and hence α, becomes positive.

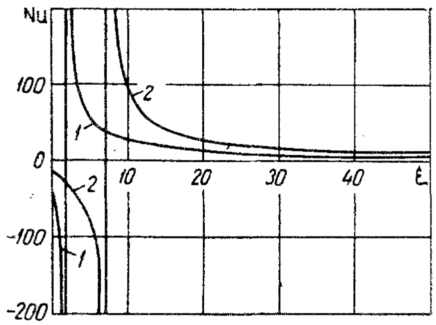

Fig. 2. Variation of local Nu over the length of a two-dimensional channel with a nonuniform temperature field at the entrance and given heat flux, $K_0 = 10$, $\beta_q = 0$, $Re = 10^4$, $Pr = 0.7$: $1 - \gamma = (-0.5)$, $2 - (-1)$.

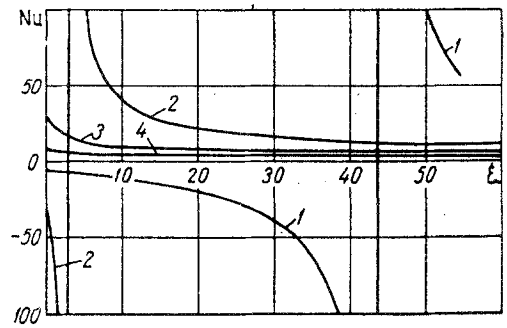

Fig. 3. Variation of local Nu over the length of a two-dimensional channel with a nonuniform temperature field at the entrance and given heat flux, $K_0 = 10$, $Re = 10^4$, $Pr = 0.7$: $1 - \beta_q = -1$, $\gamma_q = -1$; $2 - \beta_q = -1$, $\gamma_q = 1$; $3 - \beta_q = 1$, $\gamma_q = -1$; $4 - \beta_q = 1$, $\gamma_q = 1$.

Figure 3 gives the Nu distribution over the length of the channel at the upper wall, for the case $\gamma_q = -1.1$; $\beta_q = -1.1$. It may be seen from Fig. 3 that the smaller the sum $(\gamma_q + \beta_q)$, the farther from the entrance the discontinuity in the dependence of Nu on ξ. Moreover, the smaller the values of γ_q and β_q, the larger the value of Nu in the positive branch. Note that the presence of discontinuities in Nu over the length is inherently characteristic of flow with a nonuniform temperature distribution at the entrance and is related to the fact that the heat transfer coefficient γ is usually calculated from the difference between the wall temperature and the mean flow temperature. If the heat transfer coefficient α is determined from the difference between the wall temperature and the minimum fluid temperature at the entrance section, then α and Nu, thus determined, will not have discontinuities, but these quantities will be less convenient for practical calculations.

From the foregoing analysis of the effect of a nonuniform temperature field on heat transfer in a channel it follows that the presence in the entrance section of an increased temperature in the flow core and a region of low temperature near the wall leads, for a given wall temperature, to enhanced heat transfer as compared with the case of a uniform temperature field at the entrance and a given heat flux, which leads to a reduction in wall temperature. At the same time, the presence of a region of elevated temperature near the wall and reduced temperature in the flow core causes a deterioration in the heat transfer when the wall temperature is given, and an increase in wall temperature when the heat flux is given.

NOTATION

x, y – longitudinal and transverse coordinates; u, v – longitudinal and transverse velocity components; V_0 – fluid velocity in entrance section of channel; T – fluid temperature; a – thermal diffusivity; λ – thermal conductivity; 2h – height of two-dimensional channel.

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